Schwarz's Lemma & its consequences

Schwarz's Lemma :→ Let $\Delta = \{z : |z| < 1\}$ &

$$\overline{\Delta} = \{z : |z| \leq 1\}$$

Let $f : \Delta \to \overline{\Delta}$ is analytic. (i.e; $f(z)$ is analytic in $|z|<1$ & $|f(z)| \leq 1$)

Let $f(0) = 0$, Then

$$|f(z)| \leq |z|, \quad \forall\ |z| < 1.$$

Further $|f(z_0)| = |z_0|$ for an $z_0 \neq 0$ with $|z_0| < 1$ iff f is a rotation i.e;

$$f(z) = e^{i\alpha} \cdot z$$

So, $|f(z)| = |z|$, $\forall\ z$ with $|z| < 1$.

(z_0 is non-zero)

Proof :→

Evening Let $g(z) = \dfrac{f(z)}{z}$ for $z \neq 0$.

$\therefore$ g is analytic on $\Delta \setminus \{0\}$

Taylor's expansion of $f(z)$ at $z=0$ is

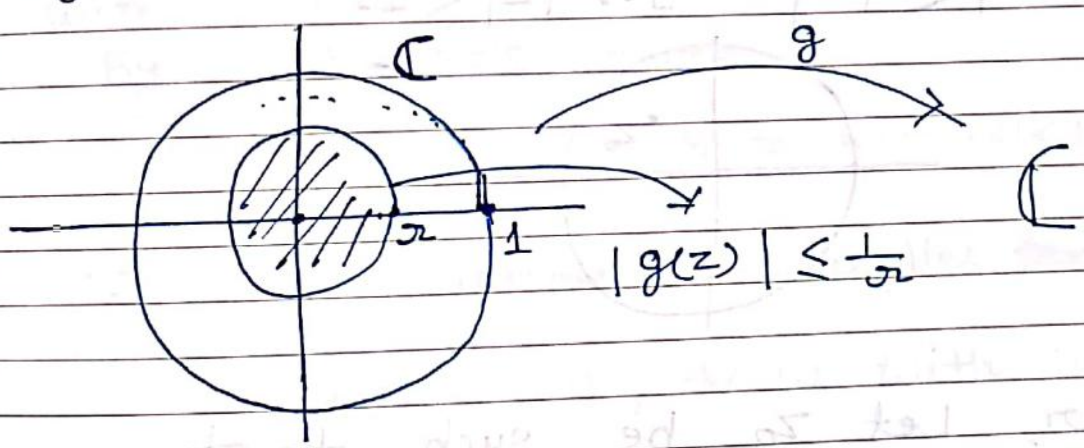
$$f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \ldots\ldots$$

$$\Rightarrow f(z) = z f'(0) + \frac{z^2}{2!} f''(0) + \cdots\cdots$$

$$\Rightarrow f(z) = z \cdot (h(z))$$

Where $h(z)$ is analytic on Δ.

As $h(z) = g(z)$ for $z \neq 0$, this shows that $g(z)$ is analytic on Δ.



On $|z| = r < 1$, $r > 0$

$$|g(z)| = \left|\frac{f(z)}{z}\right| = \frac{|f(z)|}{|z|} \leq \frac{1}{r}$$

Evening

On $|z| \leq r$

$g(z)$ has maximum on $|z| = r$.

(By the maximum principle)

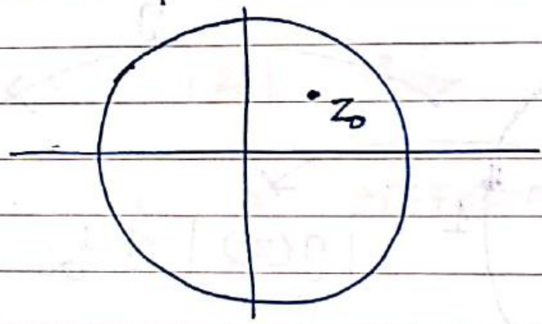
So for $|z| < r$

$$|g(z)| \leq \frac{1}{r}$$

To take $\lim_{r \to 1}$, we get

$$|g(z)| \leq 1 \quad \text{for} \quad |z| < 1$$

$$\Rightarrow |f(z)| \leq |z| \quad \text{for } |z| < 1.$$



Further, Let z_0 be such that $|z_0| < 1$, $z_0 \neq 0$ & $|f(z_0)| = |z_0|$

Then, $|g(z_0)| = 1$.

Evening

By the maximum principle, g is constant with modulus 1.

i.e; $g(z) = e^{i\alpha} \Rightarrow f(z) = z \cdot e^{i\alpha}$

i.e; f is a rotation.

Corollary :→

Any holomorphic automorphism of $\Delta = \{z : |z| < 1\}$ that fixed the origin is a rotation.

Proof :→

Let $f : \Delta \longrightarrow \Delta$ be holomorphic isomorphism with $f(0) = 0$.

By schwarz's Lemma

$$|f(z)| \leq |z|, \quad \forall z \text{ with } |z| < 1.$$

$\because$ Schwarz lemma also applies to f!

$$|f^{-1}(w)| \leq |w|, \quad \forall w \text{ with } |w| < 1.$$

$w = f(z)$ gives $|z| \leq |f(z)|$

so, $|f(z)| = |z|, \quad \forall z$ with $|z| < 1$

$\Rightarrow$ (By Schwarz's lemma)

f is a rotation.

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Schwarz lemma :→

statement :→ Let $f : \Delta \longrightarrow \overline{\Delta}$ be an analytic function, having a zero of order n at the origin.

Suppose that $|f(z)| \leq 1$, $\forall z \in \Delta$.

Then

(i) $|f(z)| \leq |z|^n$, $\forall z \in \Delta$

(ii) $|f^n(0)| \leq n!$

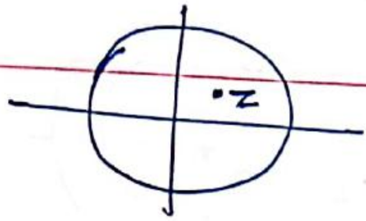
Further equality holds in (i) for some $z \neq 0$ or in (ii) if and only if $f(z)$ is of the form $f(z) = cz^n$, with $|c| = 1$.

Proof :→

Evening

Let $f : \Delta \longrightarrow \overline{\Delta}$ be analytic on Δ and have an n^{th} order zero at the origin.

Therefor $f(0) = f'(0) = \cdots\cdots = f^{(n-1)}(0) = 0$



$\because f(z)$ is analytic in $|z|<1$

$$\therefore \; f(z) = f(0) + z f'(0) + \cdots + \frac{z^n}{n!} f^n(0) + \cdots$$

$$\Rightarrow f(z) = \frac{z^n}{n!} f^n(0) + \frac{z^{n+1}}{(n+1)!} f^{n+1}(0) + \cdots \quad \text{for } z \in \Delta.$$

For $z=0$, the result is obvious.

Let $z \neq 0$ then

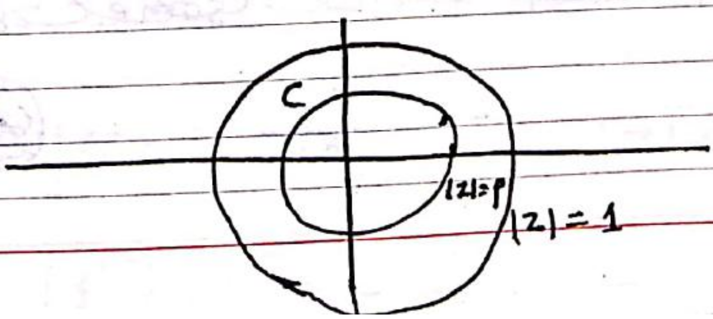
$$\frac{f(z)}{z^n} = \frac{f^{(n)}(0)}{n!} + \frac{f^{(n+1)}(0)}{(n+1)!} z + \cdots \quad 0<|z|<1.$$

The function $\frac{f(z)}{z^n}$ has a removable singularity at $z=0$.

Let us define

$$g(z) = \begin{cases} \frac{f(z)}{z^n}; & 0<|z|<1 \\ \frac{f^{(n)}(0)}{n!}, & \text{for } z=0 \end{cases}$$

Then $g(z)$ is analytic in $|z|<1$.



Let C be the circle $|z| = \rho$, where $0<\rho<1$

Then by maximum modulus principle.

$$|g(z)| \leq \max_{|z|=\rho} |g(z)| = \max_{|z|=\rho} \left|\frac{f(z)}{z^n}\right| \leq \frac{1}{\rho^n}, \quad \because |z| \leq \rho$$

The above inequality is ~~hold~~ true for all $\rho < 1$, Letting $\rho \to 1$, we find that

$$|g(z)| = \frac{|f(z)|}{|z|^n} \leq 1$$

$$\Rightarrow \quad |f(z)| \leq |z|^n \qquad \text{(i) proved}$$

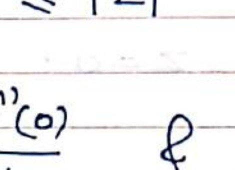
$$\because \quad g(0) = \frac{f^{(n)}(0)}{n!} \quad \&$$

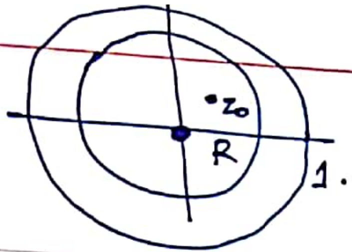
$$|g(0)| \leq 1 \Rightarrow \frac{|f^{(n)}(0)|}{n!} \leq 1$$

Evening

$$\Rightarrow |f^{(n)}(0)| \leq n! \qquad \text{(ii) proved.}$$

~~$|f(z_0)| = |z_0|^n$ for some $z_0 \neq 0$ in $0<|z_0|<R$~~





1.

If $|f(z_0)| = |z_0|^n$ for some $z_0 \neq 0$ & ~~zero~~ $0<|z_0|<R$,

then

$$g(z_0) = 1.$$

It means that $|g|$ attains its maximum at the interior point z_0.

This is only possible when $g(z)$ is constant.

say, $g(z) = c$ or, $f(z) = cz^n$ ~~for~~ for some constant with $|c| = 1$.

say $g(z) = e^{i\alpha} \Rightarrow f(z) = e^{i\alpha} z^n = cz^n$, where $|c| = 1$.

Similarly, we can prove $f^{(n)}(0) = n!$

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Corollary: → If $f(z)$ is analytic in $|z|<R$, with a zero of order n at origin. Suppose $|f(z)| \le M \ \forall z \in |z|<R$. Then

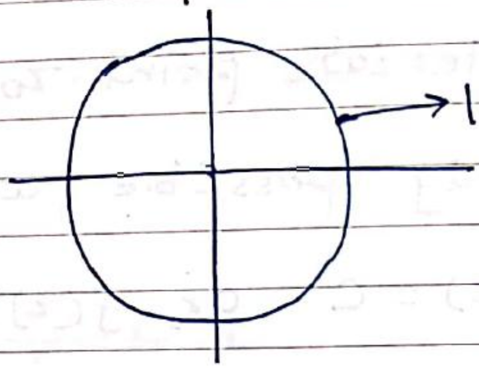
$$|f(z)| \le \frac{M|z|^n}{R^n}, \quad |z|<R \quad \text{—(I)}$$

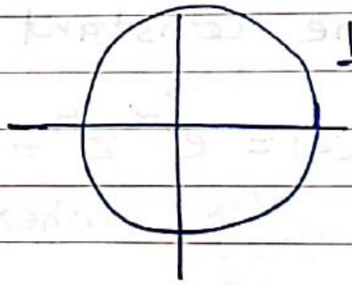
$$|f^{(n)}(0)| \le \frac{M\, n!}{R^n} \quad \text{—(II)}$$

Further, equality holds in ① for some $z \neq 0$, or in (ii) if and only if $f(z)$ is of the form

$$f(z) = \frac{Mc\,z^n}{R^n}, \quad |c| = 1.$$

<u>Proof:</u> →





$$f(z) = \frac{z^n}{n!} f^{n}(0) + \frac{z^{n+1}}{(n+1)!} f^{(n+1)}(0) + \cdots$$

~~on for |z|~~ for $z \in |z| < R$

Evening

$$\Rightarrow \frac{f(z)}{z^n} = \frac{f^n(0)}{n!} + \frac{f^{(n+1)}(0)}{(n+1)!} z + \cdots$$

$$g(z) = \begin{cases} \frac{f(z)}{z^n}, & 0 < |z| < R \\ \frac{f^{(n)}(0)}{n!} & \text{for } z = 0 \end{cases}$$

Then $g(z)$ is analytic in $|z|<R$.

Let $C: |z| = \rho$, where $0<\rho<R$

The by maximum modulus principle.

$$|g(z)| \leq \max_{|z|=\rho} |g(z)| = \max_{|z|=\rho} \left|\frac{f(z)}{z^n}\right| \leq \frac{M}{\rho^n}$$

$\because |z| \leq \rho$

The above equality is true for $\rho<R$.

Letting $\rho \to R$, we find that

$$|g(z)| = \frac{|f(z)|}{|z|^n} \leq \frac{M}{R^n}$$

$$\therefore |f(z)| \leq \frac{M|z|^n}{R^n}$$

$$\therefore |g(0)| \leq \frac{M}{R^n} \Rightarrow |f^{(n)}(0)| \leq \frac{M\,n!}{R^n}.$$

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